Aim: How do we prepare for AP Problems on limits, continuity and differentiability?

Do Now: Use the graph of f(x) to evaluate each of the following:

1. \( \lim_{{x \to 8}} f(x) \)
2. \( \lim_{{x \to 2}} f(x) \)
3. \( \lim_{{x \to -2}} f(x) \)
4. \( \lim_{{x \to -2}} f(x) \)
5. \( f(-2) \)

When does a limit exist?

Existence of a Limit: A limit exists if the left hand limit equals the right hand limit.

In graph A below, \( \lim_{{x \to 1}} f(x) = 1 \) since \( \lim_{{x \to 1^-}} f(x) = \lim_{{x \to 1^+}} f(x) = 1 \); however, in graph B, \( \lim_{{x \to 1}} f(x) = DNE \) since \( \lim_{{x \to 1^-}} f(x) = 1 \) and \( \lim_{{x \to 1^+}} f(x) = 5 \)

Graph A       Graph B

Limit as \( x \to a \). For any continuous function we can just plug in to find the limit. For example, \( \lim_{{x \to 2}} (3x^3 + 2^x) = 3 \cdot 8 + 4 = 28 \). If the function is not continuous at the given point, sometimes we can simplify before plugging in. Ex:

\[
\lim_{{x \to 2}} \frac{x^2 - 4}{x + 2} = \lim_{{x \to 2}} \frac{(x + 2)(x - 2)}{x + 2} = \lim_{{x \to 2}} (x - 2) = -2 - 2 = -4 .
\]

Vertical Asymptotes

If \( \lim_{{x \to a^*}} f(x) = \pm \infty \), then there will be a vertical asymptote at \( x = a \), as in the graph pictured to the right, where there’s a vertical asymptote at \( x = 2 \).

Limit as \( x \to \pm \infty \).

Rational function theorem: applies to rational functions. Reminder: limit as \( x \to \pm \infty \) and HA are the same thing (in both cases we consider what happens to \( y \) as \( x \to \pm \infty \)).

For a rational function \( R(x) = \frac{P(x)}{Q(x)} \) where \( P(x) \) and \( Q(x) \) are both polynomials,

1. \( \deg(Q) > \deg(P) \) [the denominator ‘dominates’]   HA: \( y = 0 \)
   \( \lim_{{x \to \infty}} R(x) = 0 \)
2. \( \deg(P) > \deg(Q) \) [the numerator ‘dominates’]    HA: none
   \( \lim_{{x \to \infty}} R(x) \text{ dne} \)
3. \( \deg(P) = \deg(Q) \) [neither ‘dominates’]       HA: \( y = \text{ratio of leading coefficients} \)
   \( \lim_{{x \to \infty}} R(x) = \frac{a}{d} \)
Otherwise, you must compare the rates of growth in each function. Know that exponential functions grow
the fastest, then polynomial functions, and then logarithmic functions. Therefore, \( \lim_{x \to \infty} \frac{\ln(x)}{x} = 0 \), since \( x \)
grows faster than \( \ln(x) \) and \( \lim_{x \to \infty} \frac{x^{100}}{e^x} = 0 \) since \( e^x \) grows faster than \( x^{100} \) and similarly \( \lim_{x \to \infty} \frac{\ln(x)}{e^x} = 0 \).

**L’Hospital’s Rule**

\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \begin{cases} \frac{0}{0} & \text{ or } \infty \infty, \text{ then } \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^\prime(x)}{g^\prime(x)} 
\end{cases}
\]

*Reminder: You can only use L’Hospital’s Rule if it’s in the above form!!!!!!!!!

Example: evaluating \( \lim_{x \to 0} \frac{\sin x}{x} \), is equivalent to evaluating \( \lim_{x \to 0} \frac{\cos x}{1} = \cos(0) = 1 \)

**Continuity:** A function is continuous if the limit exists and is equal to the function value:

\( \lim_{x \to c} f(x) = f(c) \)

The **only way** to formally demonstrate a function is continuous at a given point is
to show that the limit at that point equals the value of the function at that point.

****It is NOT sufficient to show that \( \lim_{x \to c} f(x) = \lim_{x \to c} f(x) \). All this demonstrates is
that the limit exists. You still need to show that it’s the same as the **actual value** of
the function at that point.

Example: In the graph to the right, at \( x = -1 \), the limit exists, but it’s not continuous,
since \( \lim_{x \to -1} f(x) = 5 \), but \( f(-1) = 3 \).

**Differentiability.** To show that a function is differentiable (ie smooth, ie locally linear) at a given point,
we must show that the left-hand derivative = the right-hand derivative. You must also show that they are
continuous, since all differentiable functions must be continuous.

Note: ****DIFFERENTIABILITY IMPLIES CONTINUITY!!! but continuity does NOT imply
differentiability!

**The derivative.** \( \frac{dy}{dx}, \ f'(x), \) etc. Remember, this has many different interpretations. The derivative of
a function can be thought of as:

1. The instantaneous rate of change
2. The slope of the curve
3. The slope of the tangent line
4. \( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \)
5. And the alternate form: \( f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \).
Classwork:

1. a. Given f(x) defined below, evaluate \( \lim_{x \to 2} f(x) \)
   
   \[
   f(x) = \begin{cases} 
   5 & \text{for } -5 \leq x < 2 \\
   -3 & \text{for } x = 2 \\
   x + 3 & \text{for } 2 < x < 5 
   \end{cases}
   \]

   b. Given f(x) defined below, evaluate \( \lim_{x \to 5} f(x) \)
   
   \[
   f(x) = \begin{cases} 
   4 & \text{for } x < 0 \\
   -x + 2 & \text{for } 0 \leq x < 5 \\
   3x - 16 & \text{for } x \geq 5 
   \end{cases}
   \]

2. a. Evaluate \( \lim_{x \to 2} x^2 \)
   
   b. Evaluate \( \lim_{x \to 2} \frac{x^2 - 4}{x + 2} \)

   c. Why can’t you just plug in -2 for x in problem #4 as you did in problem #3?

3. Use the graph to evaluate each limit:

   a. \( \lim_{x \to 2} f(x) \)
   
   b. \( \lim_{x \to 2} f(x) \)

4. \[
   a. \lim_{x \to \infty} \frac{3x - 7}{5x^4 - 8x + 12} = \quad b. \lim_{x \to \infty} \frac{3x^4 - 2}{5x^4 - 2x + 1} = \quad c. \lim_{x \to \infty} \frac{x^6 - 2}{10x^4 - 9x + 8} = \\
   d. \lim_{x \to \infty} \frac{(7x^2 - 2)(x^2 + x)}{5 - 2x^3 - 14x^4} = \quad e. \lim_{x \to \infty} \frac{4x^2 - 8}{16x^2 + 3x - 2} = \quad f. \lim_{x \to \infty} \frac{\sqrt{x^2 - 9}}{2x - 3}
   \]

5. How quickly does each type of function grow in comparison to each other? Put in order from slowest to fastest growing:
   
   a. Polynomial function, like \( f(x) = x^{100} \)
   b. Logarithmic function, \( f(x) = \ln(x) \)
   c. Exponential function, \( f(x) = e^x \)

   Evaluate each limit:
   
   a. \( \lim_{x \to \infty} \frac{e^x}{x^{100}} \)
   b. \( \lim_{x \to \infty} \frac{x^{100}}{e^x} \)
   c. \( \lim_{x \to \infty} \frac{x^{50}}{\ln(x)} \)
   d. \( \lim_{x \to \infty} \frac{\ln(x)}{e^x} \)
The following are the limit definitions of a derivative:

\[ \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{or} \quad \lim_{x \to a} \frac{f(x) - f(a)}{x-a} \]

a. What value is the denominator approaching in both definitions?

b. How is the instantaneous rate of change different from the average rate of change?

c. What are the other definitions of the derivative?

d. Use the limit definitions of the derivative to answer each of the following questions:

1. What is \( \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} \)?

   (A) \( \sin x \)  (B) \( \cos x \)  (C) \( -\sin x \)
   
   (D) \( -\cos x \)  (E) The limit does not exist

2. \( \lim_{\Delta x \to 0} \frac{\cos\left(\frac{\pi}{3} + \Delta x\right) - \cos\left(\frac{\pi}{3}\right)}{\Delta x} \)

   (A) \( -\frac{\sqrt{3}}{2} \)  (B) \( -\frac{1}{2} \)  (C) 0
   
   (D) \( \frac{1}{2} \)  (E) \( \frac{\sqrt{3}}{2} \)

3. \( \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} \)

   (A) \( -x^3 \)  (B) \( -3x^2 \)  (C) \( 3x^2 \)
   
   (D) \( x^3 \)  (E) The limit does not exist
6. Use the following example to explain why it is not sufficient to justify that a function \( f(x) \) is continuous at a point \( c \) if \( \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) \).

\[
f(x) = \begin{cases} 
  x + 6 & \text{for } -4 < x < -1 \\
  3 & \text{for } x = -1 \\
  -3x + 2 & \text{for } -1 < x < 3 
\end{cases}
\]

7. Give two reasons why the above function is not differentiable at \( x = -1 \).

**L'Hopital's Rule:**

Try to evaluate each of the following limits:

1. \( \lim_{x \to 0} \frac{\sin x}{x} \)
2. \( \lim_{x \to 0} \frac{x}{1 - e^x} \)

What is problematic about plugging 0 into each of the above functions in order to evaluate the limits?

**L'Hopital's Rule** says that if \( \lim_{x \to a} \frac{f(x)}{g(x)} = 0 \) or \( \infty \), then \( \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \).

**Proof:** Suppose that \( f \) and \( g \) are continuously differentiable at a real number \( c \), that \( f(c) = g(c) = 0 \), and that \( g'(c) \neq 0 \). Then

\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f(x) - f(c)}{g(x) - g(c)} = \lim_{x \to c} \frac{f'(x)}{g'(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}
\]

This follows from the difference-quotient definition of the derivative. The last equality follows from the continuity of the derivatives at \( c \). The limit in the conclusion is not indeterminate because \( g'(c) \neq 0 \).

**Examples:**

a. \( \lim_{x \to 0} \frac{\sin x}{x} \)

b. \( \lim_{x \to 0} \frac{x}{1 - e^x} \)
Classwork: Evaluate each limit. Careful...you might not be able to use l’hopital’s rule in all of these cases!

1. \( \lim_{x \to 2} \frac{3x^2 - 7x + 2}{x - 2} \)  
2. \( \lim_{x \to 2} \frac{2x + 3}{x} \)

3. \( \lim_{x \to 0} \frac{\tan x}{x} \)
4. \( \lim_{x \to 0} \frac{1 - e^x}{x - x^2} \)

5. \( \lim_{x \to 1} \frac{\ln x - x + 1}{e^x - e} \)
6. \( \lim_{x \to 0} \frac{x^2 \sin x + \cos x - 1}{x} \)

Additional Practice

1. \( \lim_{x \to 3} (x^3 - 2x + 1) \)
2. \( \lim_{x \to -1} (e^x - 2) \)
3. \( \lim_{x \to \pi} (\cos^2 x) \)
4. \( \lim_{x \to 4} \frac{x^2 - 16}{x - 4} \)
5. \( \lim_{x \to 1} \frac{x^2 - 1}{x^2 - 2x - 3} \)
6. \( \lim_{x \to 0} \frac{\sin x}{x} \)
7. \( \lim_{x \to 0} \frac{\sin^2 x}{x} \)
8. \( \lim_{x \to 0} \frac{1 - \cos^2 x}{x} \)
9. \( \lim_{x \to \infty} \frac{4x^3 - 2}{12x^6 - 1} \)
10. \( \lim_{x \to \infty} \frac{4x^3 - 2}{12x^6 - 1} \)
11. \( \lim_{x \to 0} \frac{4x^3 - 2}{12x^6 - 1} \)
12. \( \lim_{x \to 0} \frac{12x^3 - 1}{4x^3 - 2} \)
13. \( \lim_{x \to 0} \frac{12x^6 - 1}{4x^3 - 2} \)
14. \( \lim_{x \to 0} \frac{12x^6 - 1}{4x^3 - 2} \)
15. From AP practice test: \( \lim_{x \to \infty} (xe^{-x} + be^{-x}) \) where b is a constant
16. \( \lim_{x \to 0} \frac{1 + e^{-x}}{4 - e^{-2x}} \)
17. Given \( f(x) = \begin{cases} 
3x + 4, & x \leq -1 \\
-x^2 - 2x - 2, & x > -1 
\end{cases} \). Prove that the function is continuous at \( x = -1 \).

18. Given \( f(x) = \begin{cases} 
4x + 3, & x \leq 4 \\
x^2 + x - 1, & x > 4 
\end{cases} \). Prove that the function is continuous at \( x = 4 \).

19. Given \( f(x) = \begin{cases} 
x + 3, & x < 5 \\
-1, & x = 5 \\
x^2 - 4x + 3, & x > 5 
\end{cases} \). Prove that \( \lim_{x \to 5} f(x) \) exists because \( \lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) \).

Explain why the function is NOT continuous at \( x = 5 \).

20. Given \( f(x) = \begin{cases} 
3 + kx, & x < 3 \\
x^2 - 4, & x \geq 3 
\end{cases} \). Solve for \( k \) such that \( f \) is continuous at \( x = 3 \).

21. Given a differentiable function, \( f(x) = \begin{cases} 
5 + kx, & x > 1 \\
x^2 + 2x + m, & x \leq 1 
\end{cases} \), solve for \( k \) and \( m \).

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**AP Problems on Limits**

1. \( \lim_{x \to \infty} \frac{(2x - 1)(3 - x)}{(x - 1)(x + 3)} \) is
   (A) -3  (B) -2  (C) 2  (D) 3  (E) nonexistent

2. What are all horizontal asymptotes of the graph of \( y = \frac{5 + 2^x}{1 - 2^x} \) in the \( xy \)-plane?
   (A) \( y = -1 \) only
   (B) \( y = 0 \) only
   (C) \( y = 5 \) only
   (D) \( y = -1 \) and \( y = 0 \)
   (E) \( y = -1 \) and \( y = 5 \)
3. \[ \lim_{x \to 0} \frac{5x^4 + 8x^2}{3x^4 - 16x^2} \] is

(A) \( -\frac{1}{2} \)  \hspace{1cm} (B) 0  \hspace{1cm} (C) 1  \hspace{1cm} (D) \frac{5}{3}  \hspace{1cm} (E) nonexistent

4. \[ f(x) = \begin{cases} 
\frac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\
1 & \text{if } x = 2 
\end{cases} \]

Let \( f \) be the function defined above. Which of the following statements about \( f \) are true?

I. \( f \) has a limit at \( x = 2 \).
II. \( f \) is continuous at \( x = 2 \).
III. \( f \) is differentiable at \( x = 2 \).

(A) I only  
(B) II only  
(C) III only  
(D) I and II only  
(E) I, II, and III

5. The figure above shows the graph of a function \( f \) with domain \( 0 \leq x \leq 4 \). Which of the following statements are true?

I. \( \lim_{x \to 2^-} f(x) \) exists.
II. \( \lim_{x \to 2^+} f(x) \) exists.
III. \( \lim_{x \to 2} f(x) \) exists.

(A) I only  
(B) II only  
(C) I and II only  
(D) I and III only  
(E) I, II, and III
6. \[ \lim_{x \to 0} \frac{\sin x \cos x}{x} \] is

(A) \(-1\)  \hspace{1cm} (B) 0  \hspace{1cm} (C) 1  \hspace{1cm} (D) \frac{\pi}{4}  \hspace{1cm} (E) \text{ nonexistent} 

7. The graph of a function \( f \) is shown above. For which of the following values of \( c \) does \( \lim_{x \to c} f(x) = 1 \)?

(A) 0 only
(B) 0 and 3 only
(C) \(-2\) and 0 only
(D) \(-2\) and 3 only
(E) \(-2\), 0, and 3

8. \[ \lim_{x \to \infty} \frac{x^3 - 2x^2 + 3x - 4}{4x^3 - 3x^2 + 2x - 1} = \]

(A) 4  \hspace{1cm} (B) 1  \hspace{1cm} (C) \frac{1}{4}  \hspace{1cm} (D) 0  \hspace{1cm} (E) -1

\[ f(x) = \begin{cases} 
  x + 2 & \text{if } x \leq 3 \\
  4x - 7 & \text{if } x > 3
\end{cases} \]

Let \( f \) be the function given above. Which of the following statements are true about \( f \)?

I. \( \lim_{x \to 3} f(x) \) exists.
II. \( f \) is continuous at \( x = 3 \).
III. \( f \) is differentiable at \( x = 3 \).

(A) None
(B) I only
(C) II only
(D) I and II only
(E) I, II, and III
10. For which of the following does \( \lim_{x \to 4} f(x) \) exist?

I.  
\[ y \]
\[ x \]
Graph of \( f \)

II.  
\[ y \]
\[ x \]
Graph of \( f \)

III.  
\[ y \]
\[ x \]
Graph of \( f \)

(A) I only
(B) II only
(C) III only
(D) I and II only
(E) I and III only

11. The graph of the function \( f \) is shown above. Which of the following statements must be false?

(A) \( f(a) \) exists.
(B) \( f(x) \) is defined for \( 0 < x < a \).
(C) \( f \) is not continuous at \( x = a \).
(D) \( \lim_{x \to a^-} f(x) \) exists.
(E) \( \lim_{x \to a^+} f'(x) \) exists.

12. If \( f(x) = \begin{cases} \ln x & \text{for } 0 < x \leq 2 \\ x^2 \ln 2 & \text{for } 2 < x \leq 4 \end{cases} \), then \( \lim_{x \to 2} f(x) \) is

(A) \( \ln 2 \)  
(B) \( \ln 8 \)  
(C) \( \ln 16 \)  
(D) 4  
(E) nonexistent

13. If \( a \neq 0 \), then \( \lim_{x \to a} \frac{x^2 - a^2}{x - a} \) is

(A) \( \frac{1}{a^2} \)  
(B) \( \frac{1}{2a^2} \)  
(C) \( \frac{1}{6a^2} \)  
(D) 0  
(E) nonexistent
15. The graph of the function \( f \) is shown in the figure above. Which of the following statements about \( f \) is true?

(A) \( \lim_{x \to a} f(x) = \lim_{x \to b} f(x) \)
(B) \( \lim_{x \to a} f(x) = 2 \)
(C) \( \lim_{x \to b} f(x) = 2 \)
(D) \( \lim_{x \to a} f(x) = 1 \)
(E) \( \lim_{x \to a} f(x) \) does not exist.

16. \( \lim_{h \to 0} \frac{e^h - 1}{2h} \) is

(A) 0 (B) \( \frac{1}{2} \) (C) 1 (D) \( e \)  (E) nonexistent

17. \( f(x) = \begin{cases} cx + d & \text{for } x \leq 2 \\ x^2 - cx & \text{for } x > 2 \end{cases} \)

Let \( f \) be the function defined above, where \( c \) and \( d \) are constants. If \( f \) is differentiable at \( x = 2 \), what is the value of \( c + d \)?

(A) \(-4\) (B) \(-2\) (C) 0 (D) 2  (E) 4

18. The graph of a function \( f \) is shown above. At which value of \( x \) is \( f \) continuous, but not differentiable?

(A) \( a \) (B) \( b \) (C) \( c \) (D) \( d \)  (E) \( e \)
19. Let \( f \) be the function given by \( f(x) = |x| \). Which of the following statements about \( f \) are true?

I. \( f \) is continuous at \( x = 0 \).
II. \( f \) is differentiable at \( x = 0 \).
III. \( f \) has an absolute minimum at \( x = 0 \).

(A) I only  \quad (B) II only  \quad (C) III only  \quad (D) I and III only  \quad (E) II and III only

20. Let \( f \) be the function given by \( f(x) = \frac{(x-1)(x^2-4)}{x^2-a} \). For what positive values of \( a \) is \( f \) continuous for all real numbers \( x \)?

(A) None  
(B) 1 only  
(C) 2 only  
(D) 4 only  
(E) 1 and 4 only

21. Let \( f \) be a function such that \( \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = 5 \). Which of the following must be true?

I. \( f \) is continuous at \( x = 2 \).
II. \( f \) is differentiable at \( x = 2 \).
III. The derivative of \( f \) is continuous at \( x = 2 \).

(A) I only  \quad (B) II only  \quad (C) I and II only  \quad (D) I and III only  \quad (E) II and III only

22. The graph of the function \( f \) shown in the figure above has a vertical tangent at the point \((2, 0)\) and horizontal tangents at the points \((1, -1)\) and \((3, 1)\). For what values of \( x \), \(-2 < x < 4\), is \( f \) not differentiable?

(A) 0 only  \quad (B) 0 and 2 only  \quad (C) 1 and 3 only  \quad (D) 0, 1, and 3 only  \quad (E) 0, 1, 2, and 3